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Summer Term 2026
21.07.2026

Algebraic geometry 2

Exam

Family Name: _____ First name: _____

Student ID: _____ Term: _____

Degree course: Bachelor, PO ❑ 2011 ❑ 2015 ❑ 2021 Master, PO ❑ 2011 ❑ 2021

Lehramt Gymnasium: ❑ modularisiert ❑ nicht modularisiert

❑ Diplom ❑ Other: _____

Major subject: ❑ Mathematik ❑ Wirtschaftsm. ❑ Inf. ❑ Phys. ❑ Stat. ❑ _____

Minor subject: ❑ Mathematik ❑ Wirtschaftsm. ❑ Inf. ❑ Phys. ❑ Stat. ❑ _____

Credit Points to be used for ❑ Hauptfach ❑ Nebenfach (Bachelor / Master)

Please switch off your mobile phone and do not place it on the table; place your identity and student ID cards on the table so that they are clearly visible.
Please check that you have received all **five problems**.
Please do not write with the colours red or green. Write **on every page** your **family name and your first name**.
Write your solutions on the page marked with the appropriate problem number. If you run out of space, use the empty pages at the end of the examination paper ensuring that each problem is clearly marked.
Please make sure to submit only one solution for each problem; cross out everything that should not be graded.

Good luck!

1	2	3	4	5	$\sum$
/10	/7	/19	/9	/10	/55

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Problem 1. **[10 Points]**

Write for each of the following statements "true" or "false" (each correct answer gives one point, a false answer gives 0 point):

1. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. The induced homomorphism on stalks $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for every $x \in X$ if and only if $\varphi_{|U} : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for every open $U \subset X$.

2. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. If the induced homomorphism on stalks $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for every $x \in X$, then $\varphi_{|U} : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for every open $U \subset X$.

3. Let k be a field. The scheme $\operatorname{Spec} k[X, Y]/(XY)$ is reduced.

4. Let k be a field. The scheme $\operatorname{Spec} k[X, Y]/(X^2)$ is integral.

5. For every ring A and $n \geq 1$ the projective space $\mathbb{P}^n_A$ over A is a Noetherian scheme.

6. For every ring A and $n \geq 1$ the projective space $\mathbb{P}^n_A$ is a scheme of finite type over A.

7. Let $X \to S$ be a separated morphism of schemes. Let Δ be the image of the diagonal morphism $X \to X \times_S X$. Then a point $\delta \in X \times_S X$ lies in Δ if and only if $p_1(\delta) = p_2(\delta)$, where $p_1, p_2 : X \times_S X \to X$ are projections on the first and second factors.

8. Let A be a ring and $X = \operatorname{Spec} A$. Let $Z = \operatorname{Spec} A/I$ and $T = \operatorname{Spec} A/J$ be two closed subschemes of X corresponding to the ideals $I, J \subset A$. Then the scheme theoretical intersection $Z \cap T$ is isomorphic to $\operatorname{Spec} A/\sqrt{I + J}$.

9. Close immersions of schemes are stable under base change.

10. Open immersions of schemes are stable under base change.

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Problem 2. **[2+2+1+2=7 Points]**

Let k be a field and $A = k[x, y]/(x^2, xy)$. Denote by $\overline{x}$ and $\overline{y}$ the classes of x and y, respectively, in A. Consider the affine scheme $X = \operatorname{Spec} A$.

1) Show that $\overline{x} \neq 0$ and the ideal $(\overline{x})$ is prime in A. Denote by η the corresponding point in X.

2) Show that $\overline{\{\eta\}} = X$.

3) Show that the open subset $U := D(\overline{y})$ is dense in X.

4) Consider the non-zero global section $s = \overline{x} \in A = \mathcal{O}_X(X)$. Show that the restriction $s_{|U} = 0$ in $\mathcal{O}_X(U)$.

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Problem 3. **[4+1+4+6+4=19 Points]**

1) Let A be a ring and let $X = \operatorname{Spec} A$ be the corresponding affine scheme.
Let $U \subset X$ be an open dense subset and let $a \in A = \mathcal{O}_X(X)$. Assume that $a_{|U} = 0$ in $\mathcal{O}_X(U)$. Show that a is a nilpotent element in A.
Hint: Show that $U \subset V(a)$.

2) Assume that X is a reduced affine scheme. Deduce from (1) that the restriction homomorphism $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ is injective for every open dense subset $U \subset X$.

3) Let X and Y be two affine schemes, where X is reduced. Let $U \subset X$ be an open dense subset of X and let $f, g : X \to Y$ be two morphisms of schemes. Assume that $f_{|U} = g_{|U}$ as morphisms of schemes. Show that $f = g$ as morphisms of schemes.
Hint: Write $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then f and g correspond to some ring homomorphisms $\varphi, \psi : B \to A$. Show that $\varphi = \psi$ using question (2).

Let S be a scheme. Let X be a reduced scheme over S and Y a separated scheme over S. Let $f, g : X \to Y$ be two morphisms of schemes over S. Assume that $f_{|U} = g_{|U}$ (as morphisms of schemes) for some open dense subset $U \subset X$.

4) Show that $f(x) = g(x)$ for every $x \in X$.
Hint: Consider the morphism $(f, g) : X \to Y \times_S Y$ given by $f : X \to Y$ and $g : X \to Y$ (by the universal property of $Y \times_S Y$). Observe that $(f, g)_{|U} = (f, f)_{|U}$ and $(f, f) = \Delta \circ f$, where $\Delta : Y \to Y \times_S Y$ is the diagonal morphism.

5) Show that $f = g$ as morphisms.
Hint: Reduce to the case where X and Y are affine, and then use question (3).

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Problem 4. **[4+5=9 Points]**

Let X be a scheme and $x \in X$. Let $U = \operatorname{Spec} A \subset X$ be an affine open subset of X, such that $x \in U$. We have a natural ring homomorphism $A = \mathcal{O}_X(U) \to \mathcal{O}_{X,x}$, which induces a morphism $\operatorname{Spec} O_{X,x} \to U = \operatorname{Spec} A$ of affine schemes.
Let $j_x : \operatorname{Spec} O_{X,x} \to X$ be the composition $\operatorname{Spec} O_{X,x} \to U \subset X$.

1) Show that j_x does not depend on the choice of U. That is: show that for any open affine neighborhood $V = \operatorname{Spec} B$ of x the composition $\operatorname{Spec} O_{X,x} \to V \subset X$ (constructed as above) coinsides with j_x.
Hint: Use that affine open subsets form a basis of topology on X.

2) Show that the image of $j_x : \operatorname{Spec} O_{X,x} \to X$ is equal to the intersection of all open subsets of X which contain x.
Hint: Observe that $y \in X$ lies in the intersection of all open subsets of X which contain x if and only if $x \in \overline{\{y\}}$.

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Problem 5. **[4+1+5 = 10 Points]**

Let k be a field. Denote by $\mathbb{P}(k^n) = (k^{n+1}\backslash\{0\})/k^*$ the set-theoretical projective space in the classical sense and let $X = \mathbb{P}^n_k := \mathrm{Proj}(k[X_0, \ldots, X_n])$.
Recall that the following map is bijective

$$\mathbb{P}(k^n) \to X(k),\ \alpha = [\alpha_0 : \cdots : \alpha_n] \mapsto \text{ homog. prime ideal } m_\alpha = \langle (\alpha_i X_j - \alpha_j X_i)_{0 \le i,j \le n} \rangle,$$

where $X(k)$ denote the set of k-rational points of X.

1) Let F be a homogeneous polynomial in $k[X_0, \ldots, X_n]$ of degree ≥ 1.
Let $\alpha = [\alpha_0 : \cdots : \alpha_n] \in \mathbb{P}(k^n)$ and consider the corresponding k-rational point $m_\alpha \in X$.
Show that $m_\alpha \in V_+(F)$ if and only if $F(\alpha_0, \ldots, \alpha_n) = 0$.

2) Deduce from question (1) that $m_\alpha \in D_+(F)$ if and only if $F(\alpha_0, \ldots, \alpha_n) \neq 0$.

3) Assume that k is algebraically closed.
Let $\{y_1, \ldots, y_m\}$ be a finite set of k-rational points in $\mathbb{P}^n_k$. Show that there exists an affine open subset $U \subset \mathbb{P}^n_k$ containing all points $y_1, \ldots, y_m$.

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